Exam algebraic topology November 2025

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

1. Suppose $\mathcal{A}$ is an ASC and $\mathcal{B}$ is a subcomplex. Is the complement $\mathcal{A} \setminus \mathcal{B}$ also an ASC? Prove or give a counter example.
   Any non-empty ASC contains the empty set so if $\mathcal{B}$ and $\mathcal{A}$ are both non-empty and $\mathcal{A} \neq \mathcal{B}$ then the complement is a non-empty set that does not contain the empty set, hence not an ASC.

2. Give a concrete example of two connected ASC $\mathcal{A}$ and $\mathcal{B}$ with non-isomorphic fundamental groups such that $H_n(\mathcal{A}, \mathbb{F}_p) \cong H_n(\mathcal{B}, \mathbb{F}_p)$ for all n and all primes p. Prove your claims.
   We take

   $$\mathcal{A} = \mathbb{S}^2 \cup \langle\{\{0,5\},\{5,6\},\{6,0\},\{0,7\},\{7,8\},\{8,0\}\}\rangle$$

   This is a 2-sphere meeting two circles at the vertex 0. By Seifert-van Kampen the fundamental group of $\pi_1(\mathcal{A}, 0)$ is free with two generators. This can also be seen directly by picking a maximal tree by connecting all vertices directly to base point 0. That way all generators disappear except $g_{[5,6]}$ and $g_{[7,8]}$ that generate the free group. Since the dimension of $\mathcal{A}$ is two, it is shown in the lecture notes that all homology groups H_q for $q > 2$ are trivial. For the second homology group only the triangles forming the sphere contribute to the computation so $H_2(\mathcal{A}, \mathbb{F}_p) \cong \mathbb{F}_p$ imitating the computation shown in the lecture notes for the second homology of the sphere. Alternatively one can apply Mayer-Vietoris to the sphere and the other two circles. This give the same result since H_3 is trivial and so is H_1 of the intersection. The same Mayer-Vietoris argument also shows that $H_2(\mathcal{A}, \mathbb{F}_p) \cong (\mathbb{F}_p)^2$ since only the circles contribute.

   Now compare this with the torus surface $\mathcal{B}$. As shown in the lecture notes it has the same homology groups however its fundamental group is Abelian, not free.

3. Suppose you have a connected surface with Euler characteristic 0 and one boundary component. Is its orientable double cover connected?
   In the lecture notes it is shown that the orientable double cover of a connected surface is connected iff the surface is non-orientable. Our goal thus is to show that we surface $\mathcal{S}$ we are considering is not-orientable. Assuming $c \notin V(\mathcal{S})$, the cone $\Delta^c_{\partial\mathcal{S}}\mathcal{S}$ has Euler characteristic 1. This is because we are adding one vertex, and as many edges as we are adding triangles. By the classification theorem of surfaces the coned surface must be stellar equivalent to the projective plane, which is not orientable. Removing a single triangle from a non-orientable surface cannot make it orientable because any Mobius strip that was present in the original surface can be

moved so that it avoids the selected triangle. By induction therefore the original surface $\mathcal{S}$ cannot be orientable.

4. Denote by $\mathcal{A}$ the 2-skeleton of $\mathbb{S}^3$. Show that

$$\dim H_2(\mathcal{A}, \mathbb{Q}) = \dim H_2(\mathcal{A} \cup \mathbb{D}^3) + 1$$

Recall that $V(\mathbb{S}^3) = \{0, 1, 2, 3, 4\}$ and $V(\mathbb{D}^3) = \{0, 1, 2, 3\}$.
All homology groups will be computed over the field $\mathbb{Q}$. Apply the Mayer-Vietoris exact sequence to the ASC $\mathcal{A}$ and $\mathcal{B} = \mathbb{D}^3$. Then $\mathcal{A} \cap \mathbb{D}^3 = \mathbb{S}^2$ so that we obtain the exact seqnence

$$H_3(\mathcal{A} \cup \mathbb{D}^3) \to H_2(\mathbb{S}^2) \to H_2(\mathcal{A}) \oplus H_2(\mathbb{D}^3) \to H_2(\mathcal{A} \cup \mathbb{D}^3) \to H_1(\mathbb{S}^2)$$

From the lecture notes we know that $H_1(\mathbb{S}^2) = 0$ and $H_2(\mathbb{S}^2) \cong \mathbb{Q}$. By collapsing the 3-simplex we see $H_3(\mathcal{A} \cup \mathbb{D}^3) = 0$ and for the same reason $H_2(\mathbb{D}^3) = 0$. We thus end up with the exact sequence:

$$0 \to \mathbb{Q} \to H_2(\mathcal{A}) \to H_2(\mathcal{A} \cup \mathbb{D}^3) \to 0$$

which proves that $H_2(\mathcal{A} \cup \mathbb{D}^3) \cong H_2(\mathcal{A}) \oplus \mathbb{Q}$ as required.

5. Consider a path α starting and ending at point $\mathfrak{b}$ in the connected ASC $\mathcal{B}$. Denote the universal covering of $\mathcal{B}$ by $\mathcal{B}_1$. Describe the endpoint of the lift of α to $\mathcal{B}_1$ that starts at point $(\mathfrak{b}, g)$ where $g \in \pi_1(\mathcal{B}, \mathfrak{b})$.
This is the monodromy action of the fundamental group on the inverse image of $p^{-1}(\mathfrak{b})$ where p is the covering projection. For the standard covers it was shown in the lecture notes that this action corresponds to the multiplication by the fundamental group so the endpoint must be $(\mathfrak{b}, g\bar{\alpha})$.

6. Are the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g(x) = x^2$ homotopic? If yes, give a homotopy, if no prove no homotopy exists.
Yes, the straight line homotopy works here: take $H(t, x) = tf(x) + (1 - t)g(x)$. This defines a continuous function $H : [0, 1] \times \mathbb{R} \to \mathbb{R}$.

7. Define $\mathcal{A} = \langle \{\{k, k+1\} : k \in \mathbb{Z}/7\mathbb{Z}\} \rangle$ and a simplicial map $f : \mathcal{A} \to \mathbb{S}^1$ by $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and $f(3) = 2$ and $f(4) = 0$ and $f(5) = 1$ and $f(6) = 2$. If

$$\alpha = \overline{[0, 1] + [1, 2] + [2, 3] + [3, 4] + [4, 5] + [5, 6] + [6, 0]} \in C_1(\mathcal{A}, \mathbb{Q})$$

Compute $f_*(\bar{\alpha})$ in terms of the generator of $H_1(\mathbb{S}^1, \mathbb{Q})$.
It was shown in the lecture notes that the generator of $H_1(\mathbb{S}^1, \mathbb{Q})$ is precisely $\gamma = \overline{[0, 1] + [1, 2] + [2, 0]}$. The induced map is linear and sends $\overline{[a, b]}$ to $\overline{[f(a), f(b)]}$ as long as $f(a) \neq f(b)$ and to 0 otherwise. This means that $f_*(\alpha) = \overline{[0, 1] + [1, 2] + 0 + [2, 0] + [0, 1] + [1, 2] + [2, 0]} = 2\gamma$.